ALGEBRA REVIEW

In algebra, letters or variables are used to represent numbers. A **variable** is defined as a placeholder, which can take on any of several values at a given time. A **constant**, on the other hand, is a symbol which takes on only one value at a given time. A **term** is a constant, a variable, or a combination of constants and variables. For example: 7.76, $3x$, xyz, $5z/x$, $(0.99)x^2$ are terms. If a term is a combination of constants and variables, the constant part of the term is referred to as the **coefficient** of the variable. If a variable is written without a coefficient, the coefficient is assumed to be 1.

EXAMPLE

$3x^2$	y^3
coefficient: 3	coefficient: 1
variable: x	variable: y

An **expression** is a collection of one or more terms. If the number of terms is greater than 1, the expression is said to be the sum of the terms.

EXAMPLE

$$9,\ 9xy,\ 6x + x/3,\ 8yz - 2x$$

An algebraic expression consisting of only one term is called a **monomial**, of two terms is called a **binomial**, of three terms is called a **trinomial**. In general, an algebraic expression consisting of two or more terms is called a **polynomial.**

1. Operations with Polynomials

A) **Addition of polynomials** is achieved by combining like terms, terms which differ only in their numerical coefficients. E.g.,

$$P(x) = (x^2 - 3x + 5) + (4x^2 + 6x - 3)$$

Note that the parentheses are used to distinguish the polynomials.

By using the commutative and associative laws, we can rewrite $P(x)$ as:

$$P(x) = (x^2 + 4x^2) + (6x - 3x) + (5 - 3)$$

Using the distributive law, $ab + ac = a(b + c)$, yields:

$$(1 + 4)x^2 + (6 - 3)x + (5 - 3) = 5x^2 + 3x + 2$$

B) **Subtraction of two polynomials** is achieved by first changing the sign of all terms in the expression which is being subtracted and then adding this result to the other expression. E.g.,

$$(5x^2 + 4y^2 + 3z^2) - (4xy + 7y^2 - 3z^2 + 1)$$

$= 5x^2 + 4y^2 + 3z^2 - 4xy - 7y^2 + 3z^2 - 1$

$= (5x^2) + (4y^2 - 7y^2) + (3z^2 + 3z^2) - 4xy - 1$

$= (5x^2) + (-3y^2) + (6z^2) - 4xy - 1$

C) **Multiplication of two or more polynomials** is achieved by using the laws of exponents, the rules of signs, and the commutative and associative laws of multiplication. Begin by multiplying the coefficients and then multiply the variables according to the laws of exponents. E.g.,

$(y^2)(5)(6y^2)(yz)(2z^2)$

$= (1)(5)(6)(1)(2)(y^2)(y^2)(yz)(z^2)$

$= 60[(y^2)(y^2)(y)][(z)(z^2)]$

$= 60(y^5)(z^3)$

$= 60\,y^5z^3$

D) **Multiplication of a polynomial by a monomial** is achieved by multiplying each term of the polynomial by the monomial and combining the results. E.g.

$(4x^2 + 3y)(6xz^2)$

$= (4x^2)(6xz^2) + (3y)(6xz^2)$

$= 24x^3z^2 + 18xyz^2$

E) **Multiplication of a polynomial by a polynomial** is achieved by multiplying each of the terms of one polynomial by each of the terms of the other polynomial and combining the result. E.g.,

$(5y + z + 1)(y^2 + 2y)$

$[(5y)(y^2) + (5y)(2y)] + [(z)(y^2) + (z)(2y)] + [(1)(y^2) + (1)(2y)]$

$= (5y^3 + 10y^2) + (y^2z + 2yz) + (y^2 + 2y)$

$= (5y^3) + (10y^2 + y^2) + (y^2z) + (2yz) + (2y)$

$= 5y^3 + 11y^2 + y^2z + 2yz + 2y$

F) **Division of a monomial by a monomial** is achieved by first dividing the constant coefficients and the variable factors separately, and then multiplying these quotients. E.g.,

$6xyz^2 \div 2y^2z$

$= (6/2)(x/1)(y/y^2)(z^2/z)$

$= 3xy^{-1}z$

$= 3xz/y$

G) **Division of a polynomial by a polynomial** is achieved by following the given procedure called Long Division.

Step 1: The terms of both the polynomials are arranged in order of ascending or descending powers of one variable.

Step 2: The first term of the dividend is divided by the first term of the divisor which gives the first term of the quotient.

Step 3: This first term of the quotient is multiplied by the entire divisor and the result is subtracted from the dividend.

Step 4: Using the remainder obtained from Step 3 as the new dividend, Steps 2 and 3 are repeated until the remainder is zero or the degree of the remainder is less than the degree of the divisor.

Step 5: The result is written as follows:

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}} \qquad \text{divisor} \neq 0$$

e.g. $(2x^2 + x + 6) \div (x + 1)$

$$\begin{array}{r} 2x - 1 \\ x + 1 \overline{\smash{)}\, 2x^2 + \ x + 6} \\ \underline{-(2x^2 + 2x)} \\ -x + 6 \\ \underline{-(-x - 1)} \\ 7 \end{array}$$

The result is $(2x^2 + x + 6) \div (x + 1) = 2x - 1 + \dfrac{7}{x+1}$

Drill 1: Operations With Polynomials

Addition

1. $9a^2b + 3c + 2a^2b + 5c =$

(A) $19a^2bc$ (B) $11a^2b + 8c$ (C) $11a^4b^2 + 8c^2$

(D) $19a^4b^2c^2$ (E) $12a^2b + 8c^2$

2. $14m^2n^3 + 6m^2n^3 + 3m^2n^3 =$

(A) $20m^2n^3$ (B) $23m^6n^9$ (C) $23m^2n^3$

(D) $32m^6n^9$ (E) $23m^8n^{27}$

3. $3x + 2y + 16x + 3z + 6y =$

(A) $19x + 8y$ (B) $19x + 11yz$ (C) $19x + 8y + 3z$

(D) $11xy + 19xz$ (E) $30xyz$

4. $(4d^2 + 7e^3 + 12f) + (3d^2 + 6e^3 + 2f) =$

(A) $23d^2e^3f$ (B) $33d^2e^2f$ (C) $33d^4e^6f^2$

(D) $7d^2 + 13e^3 + 14f$ (E) $23d^3 + 11e^3f$

5. $3ac^2 + 2b^2c + 7ac^2 + 2ac^2 + b^2c =$

(A) $12ac^2 + 3b^2c$ (B) $14ab^2c^2$ (C) $11ac^2 + 4ab^2c$

(D) $15ab^2c^2$ (E) $15a^2b^4c^4$

Subtraction

6. $14m^2n - 6m^2n =$

(A) $20m^2n$ (B) $8m^2n$ (C) $8m$ (D) 8 (E) $8m^4n^2$

7. $3x^3y^2 - 4xz - 6x^3y^2 =$

(A) $-7x^2y^2z$ (B) $3x^3y^2 - 10x^4y^2z$ (C) $-3x^3y^2 - 4xz$

(D) $-x^2y^2z - 6x^3y^2$ (E) $-7xyz$

8. $9g^2 + 6h - 2g^2 - 5h =$

(A) $15g^2h - 7g^2h$ (B) $7g^4h^2$ (C) $11g^2 + 7h$

(D) $11g^2 - 7h^2$ (E) $7g^2 + h$

9. $7b^3 - 4c^2 - 6b^3 + 3c^2 =$

(A) $b^3 - c^2$ (B) $-11b^2 - 3c^2$ (C) $13b^3 - c$

(D) $7b - c$ (E) 0

10. $11q^2r - 4q^2r - 8q^2r =$

(A) $22q^2r$ (B) q^2r (C) $-2q^2r$

(D) $-q^2r$ (E) $2q^2r$

Multiplication

11. $5p^2t * 3p^2t =$

(A) $15p^2t$ (B) $15p^4t$ (C) $15p^4t^2$ (D) $8p^2t$ (E) $8p^4t^2$

12. $(2r + s)\ 14r =$

(A) $28rs$　(B) $28r^2 + 14sr$　(C) $16r^2 + 14rs$

(D) $28r + 14sr$　(E) $17r^2s$

13. $(4m + p)(3m - 2p) =$

(A) $12m^2 + 5mp + 2p^2$　(B) $12m^2 - 2mp + 2p^2$　(C) $7m - p$

(D) $12m - 2p$　(E) $12m^2 - 5mp - 2p^2$

14. $(2a + b)(3a^2 + ab + b^2) =$

(A) $6a^3 + 5a^2b + 3ab^2 + b^3$　(B) $5a^3 + 3ab + b^3$

(C) $6a^3 + 2a^2b + 2ab^2$　(D) $3a^2 + 2a + ab + b + b^2$

(E) $6a^3 + 3a^2b + 5ab^2 + b^3$

15. $(6t^2 + 2t + 1)\ 3t =$

(A) $9t^2 + 5t + 3$　(B) $18t^2 + 6t + 3$　(C) $9t^3 + 6t^2 + 3t$

(D) $18t^3 + 6t^2 + 3t$　(E) $12t^3 + 6t^2 + 3t$

Division

16. $(x^2 + x - 6) \div (x - 2) =$

(A) $x - 3$　(B) $x + 2$　(C) $x + 3$　(D) $x - 2$　(E) $2x + 2$

17. $24b^4c^3 \div 6b^2c =$

(A) $3b^2c^2$　(B) $4b^4c^3$　(C) $4b^3c^2$　(D) $4b^2c^2$　(E) $3b^4c^3$

18. $(3p^2 + pq - 2q^2) \div (p + q) =$

(A) $3p + 2q$　(B) $2q - 3p$　(C) $3p - q$

(D) $2q + 3p$　(E) $3p - 2q$

19. $(y^3 - 2y^2 - y + 2) \div (y - 2) =$

(A) $(y - 1)^2$　(B) $y^2 - 1$　(C) $(y + 2)(y - 1)$

(D) $(y + 1)^2$　(E) $(y + 1)(y - 2)$

20. $(m^2 + m - 14) \div (m + 4) =$

(A) $m - 2$　(B) $m - 3 + \dfrac{-2}{m + 4}$　(C) $m - 3 + \dfrac{4}{m + 4}$

(D) $m - 3$　(E) $m - 2 + \dfrac{-3}{m + 4}$

2. Simplifying Algebraic Expressions

To factor a polynomial completely is to find the prime factors of the polynomial with respect to a specified set of numbers.

The following concepts are important while factoring or simplifying expressions.

1. The factors of an algebraic expression consist of two or more algebraic expressions which when multiplied together produce the given algebraic expression.

2. **A prime factor** is a polynomial with no factors other than itself and 1. The **least common multiple (LCM)** for a set of numbers is the smallest quantity divisible by every number of the set. For algebraic expressions the least common numerical coefficients for each of the given expressions will be a factor.

3. The **greatest common factor** (GCF) for a set of numbers is the largest factor that is common to all members of the set. For algebraic expressions, the greatest common factor is the polynomial of highest degree and the largest numerical coefficient which is a factor of all the given expressions.

Some important formulae, useful for the factoring of polynomials, are listed below.

$$a(c + d) = ac + ad$$

$$(a + b)(a - b) = a^2 - b^2$$

$$(a + b)(a + b) = (a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)(a - b) = (a - b)^2 = a^2 - 2ab + b^2$$

$$(x + a)(x + b) = x^2 + (a + b)x + ab$$

$$(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd$$

$$(a + b)(c + d) = ac + bc + ad + bd$$

$$(a + b)(a + b)(a + b) = (a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a - b)(a - b)(a - b) = (a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

$$(a - b)(a^2 + ab + b^2) = a^3 - b^3$$

$$(a + b)(a^2 - ab + b^2) = a^3 + b^3$$

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$$

$$(a - b)(a^3 + a^2b + ab^2 + b^3) = a^4 - b^4$$

$$(a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4) = a^5 - b^5$$

$$(a - b)(a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5) = a^6 - b^6$$

$$(a-b)(a^{n-1}+a^{n-2}b+a^{n-3}b^2+\ldots+ab^{n-2}+b^{n-1})=a^n-b^n$$

where n is any positive integer (1, 2, 3, 4, ...).

$$(a+b)(a^{n-1}-a^{n-2}b+a^{n-3}b^2-\ldots-ab^{n-2}+b^{n-1})=a^n+b^n$$

where n is any positive odd integer (1, 3, 5, 7, ...).

The procedure for factoring an algebraic expression completely is as follows:

Step 1: First find the greatest common factor if there is any. Then examine each factor remaining for greatest common factors.

Step 2: Continue factoring the factors obtained in Step 1 until all factors other than monomial factors are prime.

EXAMPLE

Factoring $4-16x^2$,

$$4-16x^2=4(1-4x^2)=4(1+2x)(1-2x)$$

PROBLEM

Express each of the following as a single term.

(A) $3x^2+2x^2-4x^2$ (B) $5axy^2-7axy^2-3xy^2$

SOLUTION

(A) Factor x^2 in the expression.

$$3x^2+2x^2-4x^2=(3+2-4)x^2=1x^2=x^2.$$

(B) Factor xy^2 in the expression and then factor a.

$$\begin{aligned}5axy^2-7axy^2-3xy^2 &= (5a-7a-3)xy^2\\ &= [(5-7)a-3]xy^2\\ &= (-2a-3)xy^2.\end{aligned}$$

PROBLEM

Simplify $\dfrac{\frac{1}{x-1}-\frac{1}{x-2}}{\frac{1}{x-2}-\frac{1}{x-3}}$.

SOLUTION

Simplify the expression in the numerator by using the addition rule:

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$$

Notice *bd* is the Least Common Denominator, LCD. We obtain

$$\frac{(x-2)-(x-1)}{(x-1)(x-2)} = \frac{-1}{(x-1)(x-2)}$$

in the numerator.

Repeat this procedure for the expression in the denominator:

$$\frac{(x-3)-(x-2)}{(x-2)(x-3)} = \frac{-1}{(x-2)(x-3)}$$

We now have

$$\frac{\frac{-1}{(x-1)(x-2)}}{\frac{-1}{(x-2)(x-3)}},$$

which is simplified by inverting the fraction in the denominator and multiplying it by the numerator and cancelling like terms

$$\frac{-1}{(x-1)(x-2)} \cdot \frac{(x-2)(x-3)}{-1} = \frac{x-3}{x-1}.$$

Drill 2: Simplifying Algebraic Expressions

1. $16b^2 - 25z^2 =$

(A) $(4b-5z)^2$ (B) $(4b+5z)^2$ (C) $(4b-5z)(4b+5z)$

(D) $(16b-25z)^2$ (E) $(5z-4b)(5z+4b)$

2. $x^2 - 2x - 8 =$

(A) $(x-4)^2$ (B) $(x-6)(x-2)$ (C) $(x+4)(x-2)$

(D) $(x-4)(x+2)$ (E) $(x-4)(x-2)$

3. $2c^2 + 5cd - 3d^2 =$

(A) $(c-3d)(c+2d)$ (B) $(2c-d)(c+3d)$ (C) $(c-d)(2c+3d)$

(D) $(2c+d)(c+3d)$ (E) Not possible

4. $4t^3 - 20t =$

(A) $4t(t^2-5)$ (B) $4t^2(t-20)$ (C) $4t(t+4)(t-5)$

(D) $2t(2t^2-10)$ (E) Not possible

5. $x^2 + xy - 2y^2 =$

(A) $(x - 2y)(x + y)$ (B) $(x - 2y)(x - y)$ (C) $(x + 2y)(x + y)$

(D) $(x + 2y)(x - y)$ (E) Not possible

6. $5b^2 + 17bd + 6d^2 =$

(A) $(5b + d)(b + 6d)$ (B) $(5b + 2d)(b + 3d)$ (C) $(5b - 2d)(b - 3d)$

(D) $(5b - 2d)(b + 3d)$ (E) Not possible

7. $x^2 + x + 1 =$

(A) $(x + 1)^2$ (B) $(x + 2)(x - 1)$ (C) $(x - 2)(x + 1)$

(D) $(x + 1)(x - 1)$ (E) Not possible

8. $3z^3 + 6z^2 =$

(A) $3(z^3 + 2z^2)$ (B) $3z^2(z + 2)$ (C) $3z(z^2 + 2z)$

(D) $z^2(3z + 6)$ (E) $3z^2(1 + 2z)$

9. $m^2p^2 + mpg - 6g^2 =$

(A) $(mp + 3g)(mp - 2g)$ (B) $mp(mp - 2q)(mp + 3q)$

(C) $mpq(1 - 6q)$ (D) $(mp + 2q)(mp + 3q)$

(E) Not possible

10. $2h^3 + 2h^2t - 4ht^2 =$

(A) $2(h^3 - t)(h + t)$ (B) $2h(h - 2t)^2$ (C) $4h(ht - t^2)$

(D) $2h(h + t) - 4ht^2$ (E) $2h(h + 2t)(h - t)$

3. Equations

An **equation** is defined as a statement that two separate expressions are equal.

A **solution** to the equation is a number that makes the equation true when it is substituted for the variable. For example, in the equation $3x = 18$, 6 is the solution since $3(6) = 18$. Depending on the equation, there can be more than one solution. Equations with the same solutions are said to be **equivalent equations**. An equation without a solution is said to have a solution set that is the **empty** or **null** set and is represented by ϕ.

Replacing an expression within an equation by an equivalent expression will result in a new equation with solutions equivalent to the original equation.

Given the equation below

$$3x + y + x + 2y = 15$$

by combining like terms, we get,

$$3x + y + x + 2y = 4x + 3y$$

Since these two expressions are equivalent, we can substitute the simpler form into the equation to get

$$4x + 3y = 15$$

Performing the same operation to both sides of an equation by the same expression will result in a new equation that is equivalent to the original equation.

A) **Addition or subtraction**

$$y + 6 = 10$$

we can add (– 6) to both sides

$$y + 6 + (-6) = 10 + (-6)$$

to get $y + 0 = 10 - 6$; therefore $y = 4$.

B) **Multiplication or division**

$$3x = 6$$
$$3x/3 = 6/3$$
$$x = 2$$

$3x = 6$ is equivalent to $x = 2$.

C) **Raising to a power**

$$a = x^2y$$
$$a^2 = (x^2y)^2$$
$$a^2 = x^4y^2$$

This can be applied to negative and fractional powers as well. E.g.,

$$x^2 = 3y^4$$

If we raise both members to the – 2 power, we get

$$(x^2)^{-2} = (3y^4)^{-2}$$

$$\frac{1}{(x^2)^2} = \frac{1}{(3y^4)^2}$$

$$\frac{1}{x^4} = \frac{1}{9y^8}$$

If we raise both members to the $^1/_2$ power, which is the same as taking the square root, we get:

$$(x^2)^{1/2} = (3y^4)^{1/2}$$
$$x = \sqrt{3}y^2$$

D) The **reciprocal** of both members of an equation are equivalent to the original equation. Note: The reciprocal of zero is undefined.

$$\frac{2x+y}{z} = \frac{5}{2} \qquad \frac{z}{2x+y} = \frac{2}{5}$$

PROBLEM

Solve, justifying each step. $3x - 8 = 7x + 8$.

SOLUTION

	$3x - 8 = 7x + 8$
Adding 8 to both members,	$3x - 8 + 8 = 7x + 8 + 8$
Additive inverse property,	$3x + 0 = 7x + 16$
Additive identity property,	$3x = 7x + 16$
Adding $(-7x)$ to both members,	$3x - 7x = 7x + 16 - 7x$
Commuting,	$-4x = 7x - 7x + 16$
Additive inverse property,	$-4x = 0 + 16$
Additive identity property,	$-4x = 16$
Dividing both sides by -4,	$x = {}^{16}/_{-4}$
	$x = -4$

Check: Replacing x by -4 in the original equation:

$$3x - 8 = 7x + 8$$
$$3(-4) - 8 = 7(-4) + 8$$
$$-12 - 8 = -28 + 8$$
$$-20 = -20$$

Linear Equations

A linear equation with one unknown is one that can be put into the form $ax + b = 0$, where a and b are constants, $a \neq 0$.

To solve a linear equation means to transform it in the form $x = {}^{-b}/_a$.

A) If the equation has unknowns on both sides of the equality, it is convenient to put similar terms on the same sides. E.g.,

$$
\begin{aligned}
4x + 3 &= 2x + 9 \\
4x + 3 - 2x &= 2x + 9 - 2x \\
(4x - 2x) + 3 &= (2x - 2x) + 9 \\
2x + 3 &= 0 + 9 \\
2x + 3 - 3 &= 0 + 9 - 3 \\
2x &= 6 \\
{}^{2x}/_{2} &= {}^{6}/_{2} \\
x &= 3.
\end{aligned}
$$

B) If the equation appears in fractional form, it is necessary to transform it, using cross multiplication, and then repeating the same procedure as in A), we obtain:

$$\frac{3x+4}{3} = \frac{7x+2}{5}$$

By using cross multiplication we would obtain:

$$3(7x + 2) = 5(3x + 4).$$

This is equivalent to:

$$21x + 6 = 15x + 20,$$

which can be solved as in A):

$$
\begin{aligned}
21x + 6 &= 15x + 20 \\
21x - 15x + 6 &= 15x - 15x + 20 \\
6x + 6 - 6 &= 20 - 6 \\
6x &= 14 \\
x &= {}^{14}/_{6} \\
x &= {}^{7}/_{3}
\end{aligned}
$$

C) If there are radicals in the equation, it is necessary to square both sides and then apply A)

$$
\begin{aligned}
\sqrt{3x+1} &= 5 \\
\left(\sqrt{3x+1}\right)^2 &= 5^2 \\
3x + 1 &= 25 \\
3x + 1 - 1 &= 25 - 1
\end{aligned}
$$

$$3x = 24$$
$$x = {}^{24}/_3$$
$$x = 8$$

PROBLEM

Solve the equation $2(x + 3) = (3x + 5) - (x - 5)$.

SOLUTION

We transform the given equation to an equivalent equation where we can easily recognize the solution set.

$$2(x + 3) = 3x + 5 - (x - 5)$$

Distribute, $2x + 6 = 3x + 5 - x + 5$

Combine terms, $2x + 6 = 2x + 10$

Subtract $2x$ from both sides, $6 = 10$

Since $6 = 10$ is not a true statement, there is no real number which will make the original equation true. The equation is inconsistent and the solution set is ϕ, the empty set.

PROBLEM

Solve the equation $2({}^2/_3\, y + 5) + 2(y + 5) = 130$.

SOLUTION

The procedure for solving this equation is as follows:

$${}^4/_3 y + 10 + 2y + 10 = 130,$$ Distributive property

$${}^4/_3 y + 2y + 20 = 130,$$ Combining like terms

$${}^4/_3 y + 2y = 110,$$ Subtracting 20 from both sides

$${}^4/_3 y + {}^6/_3 y = 110,$$ Converting $2y$ into a fraction with denominator 3

$${}^{10}/_3 y = 110,$$ Combining like terms

$$y = 110 \cdot {}^3/_{10} = 33,$$ Dividing by ${}^{10}/_3$

Check: Replace y by 33 in the original equation,

$$2({}^2/_3(33) + 5) + 2(33 + 5) = 130$$
$$2(22 + 5) + 2(38) = 130$$
$$2(27) + 76 = 130$$

$$54 + 76 = 130$$

$$130 = 130$$

Therefore the solution to the given equation is $y = 33$.

Drill 3: Linear Equations

Solve for x:

1. $4x - 2 = 10$

(A) -1 (B) 2 (C) 3 (D) 4 (E) 6

2. $7z + 1 - z = 2z - 7$

(A) -2 (B) 0 (C) 1 (D) 2 (E) 3

3. $\frac{1}{3}b + 3 = \frac{1}{2}b$

(A) 1/2 (B) 2 (C) 3 3/5 (D) 6 (E) 18

4. $0.4p + 1 = 0.7p - 2$

(A) 0.1 (B) 2 (C) 5 (D) 10 (E) 12

5. $4(3x + 2) - 11 = 3(3x - 2)$

(A) -3 (B) -1 (C) 2 (D) 3 (E) 7

4. Two Linear Equations

Equations of the form $ax + by = c$, where a, b, c are constants and $a, b \neq 0$ are called **linear equations** with two unknown variables.

There are several ways to solve systems of linear equations in two variables:

Method 1: **Addition or subtraction** – if necessary, multiply the equations by numbers that will make the coefficients of one unknown in the resulting equations numerically equal. If the signs of equal coefficients are the same, subtract the equation, otherwise add.

The result is one equation with one unknown; we solve it and substitute the value into the other equations to find the unknown that we first eliminated.

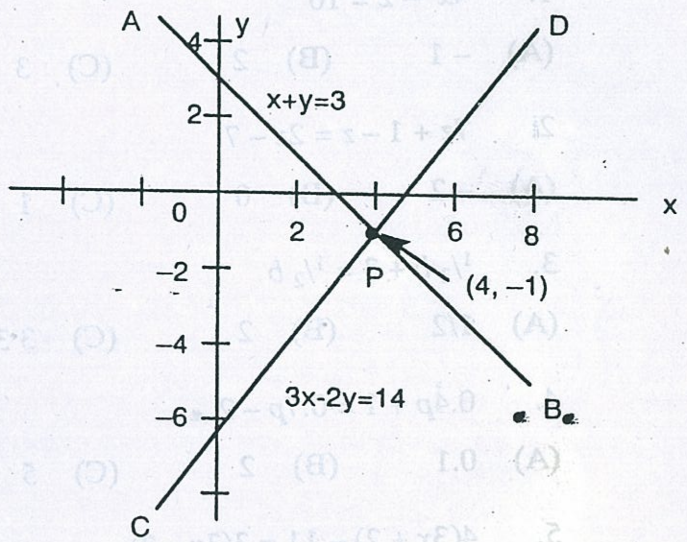
Method 2: **Substitution** – find the value of one unknown in terms of the other, substitute this value in the other equation and solve.

Method 3: **Graph** – graph both equations. The point of intersection of the drawn lines is a simultaneous solution for the equations and its coordinates correspond to the answer that would be found analytically.

If the lines are parallel they have no simultaneous solution.

Dependent equations are equations that represent the same line, therefore every point on the line of a dependent equation represents a solution. Since there is an infinite number of points on a line there is an infinite number of simultaneous solutions, for example

$$\begin{cases} 2x + y = 8 \\ 4x + 2y = 16 \end{cases}$$



The equations above are dependent, they represent the same line, all points that satisfy either of the equations are solutions of the system.

A system of linear equations is consistent if there is only one solution for the system.

A system of linear equations is inconsistent if it does not have any solutions.

Example of a consistent system. Find the point of intersection of the graphs of the equations as shown in the previous figure

$$x + y = 3,$$

$$3x - 2y = 14$$

To solve these linear equations, solve for y in terms of x. The equations will be in the form $y = mx + b$, where m is the slope and b is the intercept on the y-axis.

$x + y = 3$

$y = 3 - x$ subtract x from both sides

$3x - 2y = 14$ subtract $3x$ from both sides

$-2y = 14 - 3x$ divide by -2.

$y = -7 + {}^3/_2x$

The graphs of the linear functions, $y = 3 - x$ and $y = -7 + {}^3/_2x$ can be determined by plotting only two points. For example, for $y = 3 - x$, let $x = 0$, then $y = 3$. Let $x = 1$, then $y = 2$. The two points on this first line are (0, 3) and (1, 2). For $y = -7 +$

$^3/_2x$, let $x = 0$, then $y = -7$. Let $x = 1$, then $y = -5^1/_2$. The two points on this second line are $(0, -7)$ and $(1, -5^1/_2)$.

To find the point of intersection P of

$$x + y = 3 \quad \text{and} \quad 3x - 2y = 14,$$

solve them algebraically. Multiply the first equation by 2. Add these two equations to eliminate the variable y.

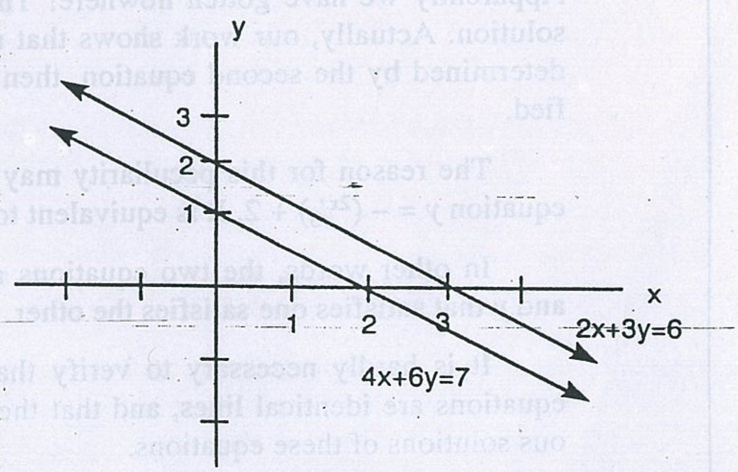
$$\begin{aligned} 2x + 2y &= 6 \\ 3x - 2y &= 14 \\ \hline 5x \quad &= 20 \end{aligned}$$

Solve for x to obtain $x = 4$. Substitute this into $y = 3 - x$ to get $y = 3 - 4 = -1$. P is $(4, -1)$. AB is the graph of the first equation, and CD is the graph of the second equation. The point of intersection P of the two graphs is the only point on both lines. The coordinates of P satisfy both equations and represent the desired solution of the problem. From the graph, P seems to be the point $(4, -1)$. These coordinates satisfy both equations, and hence are the exact coordinates of the point of intersection of the two lines.

To show that $(4, -1)$ satisfies both equations, substitute this point into both equations.

$x + y = 3$	$3x - 2y = 14$
$4 + (-1) = 3$	$3(4) - 2(-1) = 14$
$4 - 1 = 3$	$12 + 2 = 14$
$3 = 3$	$14 = 14$

Example of an inconsistent system. Solve the equations $2x + 3y = 6$ and $4x + 6y = 7$ simultaneously.



We have 2 equations in 2 unknowns,

$$2x + 3y = 6 \qquad (1)$$

and $$4x + 6y = 7 \qquad (2)$$

There are several methods to solve this problem. We have chosen to multiply each equation by a different number so that when the two equations are added, one of the variables drops out. Thus

multiplying equation (1) by 2: $\quad 4x + 6y = 12 \qquad (3)$

multiplying equation (2) by – 1: $-4x - 6y = -7$ (4)

adding equations (3) and (4): $0 = 5$

We obtain a peculiar result!

Actually, what we have shown in this case is that if there were a simultaneous solution to the given equations, then 0 would equal 5. But the conclusion is impossible; therefore there can be no simultaneous solution to these two equations, hence no point satisfying both.

The straight lines which are the graphs of these equations must be parallel if they never intersect, but not identical, which can be seen from the graph of these equations (see the accompanying diagram).

Example of a dependent system. Solve the equations $2x + 3y = 6$ and $y = -(2x/3) + 2$ simultaneously.

We have 2 equations in 2 unknowns.

$$2x + 3y = 6 \quad (1)$$

and $$y = -(2x/3) + 2 \quad (2)$$

There are several methods of solution for this problem. Since equation (2) already gives us an expression for y, we use the method of substitution. Substituting $-(2x/3) + 2$ for y in the first equation:

$$2x + 3(-2x/3 + 2) = 6$$

Distributing, $$2x - 2x + 6 = 6$$

$$6 = 6$$

Apparently we have gotten nowhere! The result $6 = 6$ is true, but indicates no solution. Actually, our work shows that no matter what real number x is, if y is determined by the second equation, then the first equation will always be satisfied.

The reason for this peculiarity may be seen if we take a closer look at the equation $y = -(2x/3) + 2$. It is equivalent to $3y = -2x + 6$, or $2x + 3y = 6$.

In other words, the two equations are equivalent. Any pair of values of x and y that satisfies one satisfies the other.

It is hardly necessary to verify that in this case the graphs of the given equations are identical lines, and that there are an infinite number of simultaneous solutions of these equations. ∞

A system of three linear equations in three unknowns is solved by eliminating one unknown from any two of the three equations and solving them. After finding two unknowns substitute them in any of the equations to find the third unknown.

PROBLEM

Solve the system

$$2x + 3y - 4z = -8 \quad (1)$$
$$x + y - 2z = -5 \quad (2)$$
$$7x - 2y + 5z = 4 \quad (3)$$

SOLUTION

We cannot eliminate any variable from two pairs of equations by a single multiplication. However, both x and z may be eliminated from equations (1) and (2) by multiplying equation 2 by -2. Then

$$2x + 3y - 4z = -8 \quad (1)$$
$$-2x - 2y + 4z = 10 \quad (4)$$

By addition, we have $y = 2$. Although we may now eliminate either x or z from another pair of equations, we can more conveniently substitute $y = 2$ in equations (2) and (3) to get two equations in two variables. Thus, making the substitution $y = 2$ in equations (2) and (3), we have

$$x - 2z = -7 \quad (5)$$
$$7x + 5z = 8 \quad (6)$$

Multiply (5) by 5 and multiply (6) by 2. Then add the two new equations. Then $x = -1$. Substitute x in either (5) or (6) to find z.

The solution of the system is $x = -1$, $y = 2$, and $z = 3$. Check by substitution.

A system of equations, as shown below, that has all constant terms $b_1, b_2, \ldots, b_n$ equal to zero is said to be a homogeneous system:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_m = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_m = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nm}x_m = b_n. \end{cases}$$

A homogeneous system always has at least one solution which is called the trivial solution that is $x_1 = 0, x_2 = 0, \ldots, x_m = 0$.

For any given homogeneous system of equations, in which the number of variables is greater than or equal to the number of equations, there are non-trivial solutions.

Two systems of linear equations are said to be equivalent if and only if they have the same solution set.

PROBLEM

Solve for x and y.

$$x + 2y = 8 \quad (1)$$

$$3x + 4y = 20 \quad (2)$$

SOLUTION

Solve equation (1) for x in terms of y:

$$x = 8 - 2y \quad (3)$$

Substitute $(8 - 2y)$ for x in (2):

$$3(8 - 2y) + 4y = 20 \quad (4)$$

Solve (4) for y as follows:

Distribute: $24 - 6y + 4y = 20$

Combine like terms and then subtract 24 from both sides:

$$24 - 2y = 20$$

$$24 - 24 - 2y = 20 - 24$$

$$-2y = -4$$

Divide both sides by -2:

$$y = 2$$

Substitute 2 for y in equation (1):

$$x + 2(2) = 8$$

$$x = 4$$

Thus, our solution is $x = 4$, $y = 2$.

Check: Substitute $x = 4$, $y = 2$ in equations (1) and (2):

$$4 + 2(2) = 8$$

$$8 = 8$$

$$3(4) + 4(2) = 20$$

$$20 = 20$$

PROBLEM

Solve algebraically:

$$4x + 2y = -1 \quad (1)$$

$$5x - 3y = 7 \quad (2)$$

SOLUTION

We arbitrarily choose to eliminate x first.

Multiply (1) by 5:	$20x + 10y = -5$	(3)
Multiply (2) by 4:	$20x - 12y = 28$	(4)
Subtract (3) – (4):	$22y = -33$	(5)
Divide (5) by 22:	$y = -{}^{33}/_{22} = -{}^{3}/_{2}$,	

To find x, substitute $y = -{}^{3}/_{2}$ in either of the original equations. If we use Eq. (1), we obtain $4x + 2(-{}^{3}/_{2}) = -1$, $4x - 3 = -1$, $4x = 2$, $x = {}^{1}/_{2}$.

The solution $({}^{1}/_{2}, -{}^{3}/_{2})$ should be checked in both equations of the given system.

Replacing $({}^{1}/_{2}, -{}^{3}/_{2})$ in Eq. (1):

$$4x + 2y = -1$$
$$4({}^{1}/_{2}) + 2(-{}^{3}/_{2}) = -1$$
$${}^{4}/_{2} - 3 = -1$$
$$2 - 3 = -1$$
$$-1 = -1$$

Replacing $({}^{1}/_{2}, -{}^{3}/_{2})$ in Eq. (2):

$$5x - 3y = 7$$
$$5({}^{1}/_{2}) - 3(-{}^{3}/_{2}) = 7$$
$${}^{5}/_{2} + {}^{9}/_{2} = 7$$
$${}^{14}/_{2} = 7$$
$$7 = 7$$

(Instead of eliminating x from the two given equations, we could have eliminated y by multiplying Eq. (1) by 3, multiplying Eq. (2) by 2, and then adding the two derived equations.)

Drill 4: Two Linear Equations

<u>DIRECTIONS</u>: Find the solution set for each pair of equations.

1. $3x + 4y = -2$
   $x - 6y = -8$

(A) $(2, -1)$ (B) $(1, -2)$ (C) $(-2, -1)$

(D) $(1, 2)$ (E) $(-2, 1)$

2. $2x + y = -10$
$-2x - 4y = 4$

(A) $(6, -2)$ (B) $(-6, 2)$ (C) $(-2, 6)$

(D) $(2, 6)$ (E) $(-6, -2)$

3. $6x + 5y = 4$
$3x - 3y = 9$

(A) $(1, -2)$ (B) $(1, 2)$ (C) $(2, -1)$

(D) $(-2, 1)$ (E) $(-1, 2)$

4. $4x + 3y = 9$
$2x - 2y = 8$

(A) $(-3, 1)$ (B) $(1, -3)$ (C) $(3, 1)$

(D) $(3, -1)$ (E) $(-1, 3)$

5. $x + y = 7$
$x - y = -3$

(A) $(5, 2)$ (B) $(-5, 2)$ (C) $(2, 5)$

(D) $(-2, 5)$ (E) $(2, -5)$

6. $5x + 6y = 4$
$3x - 2y = 1$

(A) $(3, 6)$ (B) $(1/2, 1/4)$ (C) $(-3, 6)$

(D) $(2, 4)$ (E) $(1/3, 3/2)$

7. $x - 2y = 7$
$x + y = -2$

(A) $(-2, 7)$ (B) $(3, -1)$ (C) $(-7, 2)$

(D) $(1, -3)$ (E) $(1, -2)$

8. $4x + 3y = 3$
$-2x + 6y = 3$

(A) $(1/2, 2/3)$ (B) $(-0.3, 0.6)$ (C) $(2/3, -1)$

(D) $(-0.2, 0.5)$ (E) $(0.3, 0.6)$

9. $4x - 2y = -14$
$8x + y = 7$

(A) (0, 7) (B) (2, − 7) (C) (7, 0)
(D) (− 7, 2) (E) (0, 2)

10. $6x - 3y = 1$
$-9x + 5y = -1$

(A) (1, − 1) (B) (2/3, 1) (C) (1, 2/3)
(D) (− 1, 1) (E) (2/3, − 1)

5. Quadratic Equations

A second degree equation in x of the type $ax^2 + bx + c = 0$, $a \neq 0$, a, b and c are real numbers, is called a **quadratic equation.**

To solve a quadratic equation is to find values of x which satisfy $ax^2 + bx + c = 0$. These values of x are called **solutions**, or **roots**, of the equation.

A quadratic equation has a maximum of 2 roots. Methods of solving quadratic equations:

A) **Direct solution**: Given $x^2 - 9 = 0$.

We can solve directly by isolating the variable x:

$$x^2 = 9$$

$$x = \pm 3.$$

B) **Factoring**: Given a quadratic equation $ax^2 + bx + c = 0$, $a, b, c \neq 0$, to factor means to express it as the product $a(x - r_1)(x - r_2) = 0$, where r_1 and r_2 are the two roots.

Some helpful hints to remember are:

a) $r_1 + r_2 = -{}^b/_a$.

b) $r_1 r_2 = {}^c/_a$.

Given $x^2 - 5x + 4 = 0$.

Since $r_1 + r_2 = -{}^b/_a = -{}^{(-5)}/_1 = 5$, some possible solutions are (3, 2), (4, 1) and (5, 0). Also $r_1 r_2 = {}^c/_a = {}^4/_1 = 4$; this equation is satisfied only by the second pair, so $r_1 = 4$, $r_2 = 1$ and the factored form is $(x - 4)(x - 1) = 0$.

If the coefficient of x^2 is not 1, it is necessary to divide the equation by this coefficient and then factor.

Given $2x^2 - 12x + 16 = 0$

Dividing by 2, we obtain:

$$x^2 - 6x + 8 = 0$$

Since $r_1 + r_2 = -{}^b/_a = 6$, possible solutions are (6, 0), (5, 1), (4, 2), (3, 3). Also $r_1 r_2 = 8$, so the only possible answer is (4, 2) and the expression $x^2 - 6x + 8 = 0$ can be factored as $(x - 4)(x - 2)$.

C) **Completing the Squares:**

If it is difficult to factor the quadratic equation using the previous method, we can complete the squares.

Given $x^2 - 12x + 8 = 0$.

We know that the two roots added up should be 12 because $r_1 + r_2 = -{}^b/_a = -{}^{(-12)}/_1 = 12$. Possible roots are (12, 0), (11, 1), (10, 2), (9, 3), (8,4), (7,5), (6, 6).

But none of these satisfy $r_1 r_2 = 8$, so we cannot use (B).

To complete the square, it is necessary to isolate the constant term,

$$x^2 - 12x = -8.$$

Then take $^1/_2$ coefficient of x, square it and add to both sides.

$$x^2 - 12x + \left(\frac{-12}{2}\right)^2 = -8 + \left(\frac{-12}{2}\right)^2$$

$$x^2 - 12x + 36 = -8 + 36 = 28.$$

Now we can use the previous method to factor the left side: $r_1 + r_2 = 12$, $r_1 r_2 = 36$ is satisfied by the pair (6, 6), so we have:

$$(x - 6)^2 = 28.$$

Now extract the root of both sides and solve for x.

$$(x - 6) = \pm\sqrt{28} = \pm 2\sqrt{7}$$

$$x = \pm 2\sqrt{7} + 6$$

So the roots are:

$$x = 2\sqrt{7} + 6, \quad x = -2\sqrt{7} + 6.$$

PROBLEM

Solve the equation $x^2 + 8x + 15 = 0$.

SOLUTION

Since $(x + a)(x + b) = x^2 + bx + ax + ab = x^2 + (a + b)x + ab$, we may factor the given equation, $0 = x^2 + 8x + 15$, replacing $a + b$ by 8 and ab by 15. Thus,

$a + b = 8,\quad \text{and}\quad ab = 15.$

We want the two numbers a and b whose sum is 8 and whose product is 15. We check all pairs of numbers whose product is 15:

(a) $1 \cdot 15 = 15$; thus $a = 1$, $b = 15$ and $ab = 15$.

$1 + 15 = 16$, therefore we reject these values because $a + b \neq 8$.

(b) $3 \cdot 5 = 15$, thus $a = 3$, $b = 5$, and $ab = 15$.

$3 + 5 = 8$. Therefore $a + b = 8$, and we accept these values.

Hence $x^2 + 8x + 15 = 0$ is equivalent to

$$0 = x^2 + (3 + 5)x + 3 \cdot 5 = (x + 3)(x + 5)$$

Hence, $\quad x + 5 = 0$ or $x + 3 = 0$

since the product of these two numbers is zero, one of the numbers must be zero. Hence, $x = -5$, or $x = -3$, and the solution set is $x = \{-5, -3\}$.

The student should note that $x = -5$ or $x = -3$. We are certainly not making the statement, that $x = -5$, and $x = -3$. Also, the student should check that both these numbers do actually satisfy the given equations and hence are solutions.

Check: Replacing x by (-5) in the original equation:

$$x^2 + 8x + 15 = 0$$

$$(-5)^2 + 8(-5) + 15 = 0$$

$$25 - 40 + 15 = 0$$

$$-15 + 15 = 0$$

$$0 = 0$$

Replacing x by (-3) in the original equation:

$$x^2 + 8x + 15 = 0$$

$$(-3)^2 + 8(-3) + 15 = 0$$

$$9 - 24 + 15 = 0$$

$$-15 + 15 = 0$$

$$0 = 0.$$

PROBLEM

Solve the following equations by factoring.

(a) $2x^2 + 3x = 0$ (c) $z^2 - 2z - 3 = 0$

(b) $y^2 - 2y - 3 = y - 3$ (d) $2m^2 - 11m - 6 = 0$

SOLUTION

(a) $2x^2 + 3x = 0$. Factoring out the common factor of x from the left side of the given equation,

$$x(2x + 3) = 0.$$

Whenever a product $ab = 0$, where a and b are any two numbers, either $a = 0$ or $b = 0$. Then, either

$$x = 0 \quad \text{or} \quad 2x + 3 = 0$$
$$2x = -3$$
$$x = -3/2$$

Hence, the solution set to the original equation $2x^2 + 3x = 0$ is: $\{-3/2, 0\}$.

(b) $y^2 - 2y - 3 = y - 3$. Subtract $(y - 3)$ from both sides of the given equation:

$$y^2 - 2y - 3 - (y - 3) = y - 3 - (y - 3)$$
$$y^2 - 2y - 3 - y + 3 = y - 3 - y + 3$$
$$y^2 - 2y - 3 - y + 3 = y - 3 - y + 3$$
$$y^2 - 3y = 0.$$

Factor out a common factor of y from the left side of this equation:

$$y(y - 3) = 0.$$

Thus, $y = 0$ or $y - 3 = 0$, $y = 3$.

Therefore, the solution set to the original equation $y^2 - 2y - 3 = y - 3$ is: $\{0,3\}$.

(c) $z^2 - 2z - 3 = 0$. Factor the original equation into a product of two polynomials:

$$z^2 - 2z - 3 = (z - 3)(z + 1) = 0$$

Hence,

$$(z - 3)(z + 1) = 0; \text{ and } z - 3 = 0 \text{ or } z + 1 = 0$$
$$z = 3 \qquad z = -1$$

Therefore, the solution set to the original equation $z^2 - 2z - 3 = 0$ is: $\{-1, 3\}$.

(d) $2m^2 - 11m - 6 = 0$. Factor the original equation into a product of two polynomials:

$$2m^2 - 11m - 6 = (2m + 1)(m - 6) = 0$$

Thus,

$$2m + 1 = 0 \quad \text{or} \quad m - 6 = 0$$

$2m = -1$ $\quad\quad m = 6$

$m = -^1/_2$

Therefore, the solution set to the original equation $2m^2 - 11m - 6 = 0$ is $\{-^1/_2, 6\}$.

Drill 5: Quadratic Equations

DIRECTIONS: Solve for all values of x.

1. $x^2 - 2x - 8 = 0$

(A) 4 and – 2 (B) 4 and 8 (C) 4
(D) – 2 and 8 (E) – 2

2. $x^2 + 2x - 3 = 0$

(A) – 3 and 2 (B) 2 and 1 (C) 3 and 1
(D) – 3 and 1 (E) – 3

3. $x^2 - 7x = -10$

(A) – 3 and 5 (B) 2 and 5 (C) 2
(D) – 2 and – 5 (E) 5

4. $x^2 - 8x + 16 = 0$

(A) 8 and 2 (B) 1 and 16 (C) 4
(D) – 2 and 4 (E) 4 and – 4

5. $3x^2 + 3x = 6$

(A) 3 and – 6 (B) 2 and 3 (C) – 3 and 2
(D) 1 and – 3 (E) 1 and – 2

6. $x^2 + 7x = 0$

(A) 7 (B) 0 and – 7 (C) – 7
(D) 0 and 7 (E) 0

7. $x^2 - 25 = 0$

(A) 5 (B) 5 and – 5 (C) 15 and 10
(D) – 5 and 10 (E) – 5

8. $2x^2 + 4x = 16$

(A) 2 and – 2 (B) 8 and – 2 (C) 4 and 8

(D) 2 and – 4 (E) 2 and 4

9. $6x^2 - x - 2 = 0$

(A) 2 and 3 (B) 1/2 and 1/3 (C) – 1/2 and 2/3

(D) 2/3 and 3 (E) 2 and – 1/3

10. $12x^2 + 5x = 3$

(A) 1/3 and – 1/4 (B) 4 and – 3 (C) 4 and 1/6

(D) 1/3 and – 4 (E) – 3/4 and 1/3

6. Absolute Value Equations

The absolute value of a, $|a|$, is defined as:

$$|a| = a \text{ when } a > 0,\ |a| = -a \text{ when } a < 0,\ |a| = 0 \text{ when } a = 0.$$

When the definition of absolute value is applied to an equation, the quantity within the absolute value symbol is considered to have two values. This value can be either positive or negative before the absolute value is taken. As a result, each absolute value equation actually contains two separate equations.

When evaluating equations containing absolute values, proceed as follows:

EXAMPLE

$|5 - 3x| = 7$ is valid if either

$5 - 3x = 7$	or	$5 - 3x = -7$
$-3x = 2$		$-3x = -12$
$x = -2/3$		$x = 4$

The solution set is therefore $x = (-2/3, 4)$.

Remember, the absolute value of a number cannot be negative. So, for the equation $|5x + 4| = -3$, there would be no solution.

EXAMPLE

Solve for x in $|2x - 6| = |4 - 5x|$

There are four possibilities here. $2x - 6$ and $4 - 5x$ can be either positive or negative. Therefore,

$$2x - 6 = 4 - 5x \quad (1)$$
$$-(2x - 6) = 4 - 5x \quad (2)$$
$$2x - 6 = -(4 - 5x) \quad (3)$$
$$-(2x - 6) = -(4 - 5x) \quad (4)$$

Equations (2) and (3) result in the same solution, as do equations (1) and (4). Therefore, it is necessary to solve only for equations (1) and (2). This gives:

$$2x\ 6 = 4 - 5x \qquad \text{or} \qquad -(2x - 6) = 4 - 5x$$
$$7x = 10 \qquad\qquad -2x + 6 = 4 - 5x$$
$$3x = -2$$
$$x = 10/7 \qquad\qquad x = -2/3$$

The solution set is (10/7, – 2/3).

Drill 6: Absolute Value Equations

1. $|4x - 2| = 6$

(A) – 2 and – 1 (B) – 1 and 2 (C) 2
(D) 1/2 and – 2 (E) No solution

2. $|3 - 1/2y| = -7$

(A) – 8 and 20 (B) 8 and – 20 (C) 2 and – 5
(D) 4 and – 2 (E) No solution

3. $2|x + 7| = 12$

(A) – 13 and – 1 (B) – 6 and 6 (C) – 1 and 13
(D) 6 and – 13 (E) No solution

4. $|5x| - 7 = 3$

(A) 2 and 4 (B) 4/5 and 3 (C) – 2 and 2
(D) 2 (E) No solution

5. $\left|\frac{3}{4}m\right| = 9$

(A) 24 and – 16 (B) 4/27 and – 4/3 (C) 4/3 and 12
(D) – 12 and 12 (E) No solution

7. Inequalities

An inequality is a statement where the value of one quantity or expression is greater than (>), less than (<), greater than or equal to (≥), less than or equal to (≤), or not equal to (≠) that of another.

EXAMPLE

$5 > 4.$

The expression above means that the value of 5 is greater than the value of 4.

A **conditional inequality** is an inequality whose validity depends on the values of the variables in the sentence. That is, certain values of the variables will make the sentence true, and others will make it false. $3 - y > 3 + y$ is a conditional inequality for the set of real numbers, since it is true for any replacement less than zero and false for all others.

$x + 5 > x + 2$ is an **absolute inequality** for the set of real numbers, meaning that for any real value x, the expression on the left is greater than the expression on the right.

$5y < 2y + y$ is inconsistent for the set of non-negative real numbers. For any y greater than 0 the sentence is always false. A sentence is inconsistent if it is always false when its variables assume allowable values.

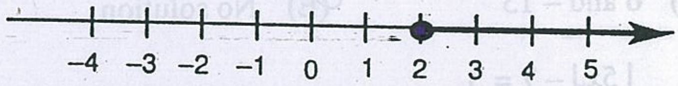
The solution of a given inequality in one variable x consists of all values of x for which the inequality is true.

The graph of an inequality in one variable is represented by either a ray or a line segment on the real number line.

The endpoint is not a solution if the variable is strictly less than or greater than a particular value.

EXAMPLE

$x > 2$

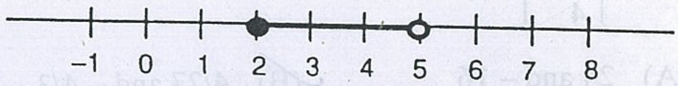


2 is not a solution and should be represented as shown.

The endpoint is a solution if the variable is either (1) less than or equal to or (2) greater than or equal to, a particular value.

EXAMPLE

$5 > x \geq 2$



In this case 2 is the solution and should be represented as shown.

Properties of Inequalities

If x and y are real numbers then one and only one of the following statements is true.

$x > y, x = y$ or $x < y$.

This is the order property of real numbers.

If a, b and c are real numbers:

A) If $a < b$ and $b < c$ then $a < c$.

B) If $a > b$ and $b > c$ then $a > c$.

This is the transitive property of inequalities.

If a, b and c are real numbers and $a > b$ then $a + c > b + c$ and $a - c > b - c$. This is the **addition property of inequality**.

Two inequalities are said to have the same sense if their signs of inequality point in the same direction.

The sense of an inequality remains the same if both sides are multiplied or divided by the same positive real number.

EXAMPLE

$4 > 3$

If we multiply both sides by 5 we will obtain:

$4 \times 5 > 3 \times 5$

$20 > 15$

The sense of the inequality does not change.

The sense of an inequality becomes opposite if each side is multiplied or divided by the same negative real number.

EXAMPLE

$4 > 3$

If we multiply both sides by -5 we would obtain:

$4 \times -5 < 3 \times -5$

$-20 < -15$

The sense of the inequality becomes opposite.

If $a > b$ and a, b and n are positive real numbers, then:

$a^n > b^n$ and $a^{-n} < b^{-n}$

If $x > y$ and $q > p$ then $x + q > y + p$.

If $x > y > 0$ and $q > p > 0$ then $xq > yp$.

Inequalities that have the same solution set are called **equivalent inequalities.**

PROBLEM

Solve the inequality $2x + 5 > 9$.

SOLUTION

$2x + 5 + (-5) > 9 + (-5)$.	Adding -5 to both sides.
$2x + 0 > 9 + (-5)$	Additive inverse property
$2x > 9 + (-5)$	Additive identity property
$2x > 4$	Combining terms
$\frac{1}{2}(2x) > \frac{1}{2} \cdot 4$	Multiplying both sides by $\frac{1}{2}$.
$x > 2$	

The solution set is

$$X = \{x \mid 2x + 5 > 9\}$$
$$= \{x \mid x > 2\}$$

(that is all x, such that x is greater than 2).

PROBLEM

Solve the inequality $4x + 3 < 6x + 8$.

SOLUTION

In order to solve the inequality $4x + 3 < 6x + 8$, we must find all values of x which make it true. Thus, we wish to obtain x alone on one side of the inequality.

Add -3 to both sides:

$$\begin{array}{r} 4x + 3 < 6x + 8 \\ -3 \quad\quad -3 \\ \hline 4x < 6x + 5 \end{array}$$

Add $-6x$ to both sides:

$$\begin{array}{rr} 4x < & 6x + 5 \\ -6x \quad & -6x \quad\quad \\ \hline -2x < & 5 \end{array}$$

In order to obtain x alone we must divide both sides by (-2). Recall that dividing an inequality by a negative number reverses the inequality sign, hence

$$\frac{-2x}{-2} > \frac{5}{-2}$$

Cancelling $-^2/_{-2}$ we obtain, $x > -^5/_2$.

Thus, our solution is $\{x : x > -^5/_2\}$ (the set of all x such that x is greater than $-^5/_2$).

Drill 7: Inequalities

DIRECTIONS: Find the solution set for each inequality

1. $3m + 2 < 7$

(A) $m \geq ^5/_3$ (B) $m \leq 2$ (C) $m < 2$

(D) $m > 2$ (E) $m < ^5/_3$

2. $^1/_2 x - 3 \leq 1$

(A) $-4 \leq x \leq 8$ (B) $x \geq -8$ (C) $x \leq 8$

(D) $2 \leq x \leq 8$ (E) $x \geq 8$

3. $-3p + 1 \geq 16$

(A) $p \geq -5$ (B) $p \geq \frac{-17}{3}$ (C) $p \leq \frac{-17}{3}$

(D) $p \leq -5$ (E) $p \geq 5$

4. $-6 < ^2/_3 r + 6 \leq 2$

(A) $-6 < r \leq -3$ (B) $-18 < r \leq -6$ (C) $r \geq -6$

(D) $-2 < r \leq -^4/_3$ (E) $r \leq -6$

5. $0 < 2 - y < 6$

(A) $-4 < y < 2$ (B) $-4 < y < 0$ (C) $-4 < y < -2$

(D) $-2 < y < 4$ (E) $0 < y < 4$

8. Ratios and Proportions

The ratio of two numbers x and y written $x : y$ is the fraction x / y where $y \neq 0$. A ratio compares x to y by dividing one by the other. Therefore, in order to compare ratios, simply compare the fractions.

A proportion is an equality of two ratios. The laws of proportion are listed below:

If $a/b = c/d$, then

(A) $ad = bc$

(B) $b/a = d/c$

(C) $a/c = b/d$

(D) $(a + b)/b = (c + d)/d$

(E) $(a - b)/b = (c - d)/d$

Given a proportion $a : b = c : d$, then a and d are called extremes, b and c are called the means and d is called the fourth proportion to a, b, and c.

PROBLEM

Solve the proportion $\frac{x+1}{4} = \frac{15}{12}$.

SOLUTION

Cross multiply to determine x; that is, multiply the numerator of the first fraction by the denominator of the second, and equate this to the product of the numerator of the second and the denominator of the first.

$$(x + 1)\,12 = 4 \cdot 15$$

$$12x + 12 = 60$$

$$x = 4.$$

PROBLEM

Find the ratios of $x : y : z$ from the equations

$7x = 4y + 8z, \quad 3z = 12x + 11y.$

SOLUTION

By transposition we have

$$7x - 4y - 8z = 0$$

$$12x + 11y - 3z = 0.$$

To obtain the ratio of $x : y$ we convert the given system into an equation in terms of just x and y. z may be eliminated as follows: Multiply each term of the first equation by 3, and each term of the second equation by 8, and then subtract the second equation from the first. We thus obtain:

$$\begin{array}{r} 21x - 12y - 24z = 0 \\ -(96x + 88y - 24z = 0) \\ \hline -75x - 100y \quad = 0 \end{array}$$

Dividing each term of the last equation by 25 we obtain

$$-3x - 4y = 0$$

or,

$$-3x = 4y.$$

Dividing both sides of this equation by 4, and by − 3, we have the proportion:

$$\frac{x}{4} = \frac{y}{-3}$$

We are now interested in obtaining the ratio of $y : z$. To do this we convert the given system of equations into an equation in terms of just y and z, by eliminating x as follows: Multiply each term of the first equation by 12, and each term of the second equation by 7, and then subtract the second equation from the first. We thus obtain:

$$\begin{array}{r} 84x - 48y - 96z = 0 \\ -(84x + 77y - 21z = 0) \\ \hline -125y - 75z = 0. \end{array}$$

Dividing each term of the last equation by 25 we obtain

$$-5y - 3z = 0$$

or,

$$-3z = 5y.$$

Dividing both sides of this equation by 5, and by − 3, we have the proportion:

$$\frac{z}{5} = \frac{y}{-3}.$$

From this result and our previous result we obtain:

$$\frac{x}{4} = \frac{y}{-3} = \frac{z}{5}$$

as the desired ratios.

Drill 8: Ratios and Proportions

1. Solve for n : $\frac{4}{n} = \frac{8}{5}$

(A) 10 (B) 8 (C) 6 (D) 2.5 (E) 2

2. Solve for n: $\frac{2}{3} = \frac{n}{72}$

(A) 12 (B) 48 (C) 64 (D) 56 (E) 24

3. Solve for n: $n : 12 = 3 : 4$.

(A) 8 (B) 1 (C) 9 (D) 4 (E) 10

4. Four out of every five students at West High take a mathematics course. If the enrollment at West is 785, how many students take mathematics?

(A) 628 (B) 157 (C) 705 (D) 655 (E) 247

5. At a factory, three out of every 1,000 parts produced are defective. In a day, the factory can produce 25,000 parts. How many of these parts would be defective?

(A) 7 (B) 75 (C) 750 (D) 7,500 (E) 75,000

6. A summer league softball team won 28 out of the 32 games they played. What is the ratio of games won to games played?

(A) 4 : 5 (B) 3 : 4 (C) 7 : 8 (D) 2 : 3 (E) 1 : 8

7. A class of 24 students contains 16 males. What is the ratio of females to males?

(A) 1 : 2 (B) 2 : 1 (C) 2 : 3 (D) 3 : 1 (E) 3 : 2

8. A family has a monthly income of $1,250, but they spend $450 a month on rent. What is the ratio of the amount of income to the amount paid for rent?

(A) 16 : 25 (B) 25 : 9 (C) 25 : 16 (D) 9 : 25 (E) 36 : 100

9. A student attends classes 7.5 hours a day and works a part-time job for 3.5 hours a day. She knows she must get 7 hours of sleep a night. Write the ratio of the number of free hours in this student's day to the total number of hours in a day.

(A) 1 : 3 (B) 4 : 3 (C) 8 : 24 (D) 1 : 4 (E) 5 : 12

10. In a survey by mail, 30 out of 750 questionnaires were returned. Write the ratio of questionnaires returned to questionnaires mailed (write in simplest form).

(A) 30 : 750 (B) 24 : 25 (C) 3 : 75 (D) 1 : 4 (E) 1 : 25

ALGEBRA DRILLS

ANSWER KEY

Drill 1—Operations With Polynomials

1.	(B)	6.	(B)	11.	(C)	16.	(C)
2.	(C)	7.	(C)	12.	(B)	17.	(D)
3.	(C)	8.	(E)	13.	(E)	18.	(E)
4.	(D)	9.	(A)	14.	(A)	19.	(B)
5.	(A)	10.	(D)	15.	(D)	20.	(B)

Drill 2—Simplifying Algebraic Expressions

1.	(C)	6.	(B)
2.	(D)	7.	(E)
3.	(B)	8.	(B)
4.	(A)	9.	(A)
5.	(D)	10.	(E)

Drill 3—Linear Equations

1.	(C)
2.	(A)
3.	(E)
4.	(D)
5.	(B)

Drill 4—Two Linear Equations

1.	(E)	6.	(B)
2.	(B)	7.	(D)
3.	(A)	8.	(E)
4.	(D)	9.	(A)
5.	(C)	10.	(B)

Drill 5—Quadratic Equations

1.	(A)	6.	(B)
2.	(D)	7.	(B)
3.	(B)	8.	(D)
4.	(C)	9.	(C)
5.	(E)	10.	(E)

Drill 6—Absolute Value Equations

1.	(B)	4.	(C)
2.	(E)	5.	(D)
3.	(A)		

Drill 7—Inequalities

1.	(E)	4.	(B)
2.	(C)	5.	(A)
3.	(D)		

Drill 8—Ratios and Proportions

1.	(D)	4.	(A)	7.	(A)	10.	(E)
2.	(B)	5.	(B)	8.	(B)		
3.	(C)	6.	(C)	9.	(D)		

GLOSSARY: ALGEBRA

Abscissa

The x (horizontal) value of a point in the Cartesian plane.

Absolute Inequality

An inequality that is true for all values of all variables.

Absolute Value

The numerical value of a number without regard to sign (i.e., it is always nonnegative).

Additive Property of Inequalities

The property that states that, if $a < b$, then $(a+c) < (b + c)$.

Algebra

The study of using letters to represent arbitrary numbers for the purpose of stating results in greater generality.

Binomial

An expression consisting of two terms.

Cartesian Coordinates

Ordered pairs of numbers assigned to points in the Cartesian plane.

Cartesian Plane

A two-dimensional grid used for graphically placing ordered pairs of reals relative to one another.

Closed Interval

An interval that contains its endpoints.

Coefficient

The number that multiplies a variable. If a variable is written alone, then the coefficient is assumed to be 1.

Completing the Squares

A method of factoring quadratic equations by adding and subtracting an appropriate quantity.

Complex Numbers

The sum of a real number and an imaginary number, i.e., $a + bi$, where a and b are real and $i^2 = -1$.

Conditional Inequality

An inequality whose validity depends on the values of one or more variables.

Constant

A variable that takes on only one fixed value.

Coordinate Axes

Two perpendicular lines (the x-axis and the y-axis) used for placing ordered pairs of reals relative to one another.

Dependent Equation

Equations with the same solution sets.

Equal Sets

Sets that contain the same members.

Equations

The statement that two expressions are equal. These expressions may depend on one or more variables.

Equivalent Equations

Equations with the same solution sets.

Equivalent Inequalities

Inequalities with the same solution sets.

Equivalent Sets

Sets that have the same number of elements.

Expression

A collection of terms joined by addition, subtraction, multiplication, or division. When all variables are given numerical values, the expression also has a numerical value.

Factoring

Finding a set of expressions whose product is the given expression. To factor the quadratic equation $ax^2 + bx + c = 0$ is to find r_1, r_2 such that $a(x - r_1)(x - r_2) = 0$.

Greatest Common Factor

The largest number that has the property that, when any of a group of numbers is divided by it, the result is an integer.

Half-Open Interval

An interval that contains one of its endpoints.

Imaginary Numbers

Numbers of the form bi, where b is real and $i^2 = -1$.

Inconsistent Inequality

An inequality that is false for all values of all variables.

Inequality

The statement that two expressions are not equal or that one exceeds the other in numerical value or that one is at least as large or small in numerical value than the other.

Least Common Multiple

The smallest number that results in an integer regardless of which of a given set of numbers it is divided by.

Linear Equation

An equation in which the exponent of each variable is 1. When plotted, the solution set forms a straight line in the Cartesian plane.

Monomial

An expression consisting of one term.

Null Set

A set with no elements or members.

Open Interval

An interval that contains neither of its endpoints.

Ordered Pair

A pair of elements of the form (x, y), in which the order is specified. Thus (x, y) in general is *not* the same as (y, x).

Ordinate

The y (vertical) value of a point in the Cartesian plane.

Origin

The intersection of the two coordinate axes. The origin corresponds to the ordered pair (0, 0).

Polynomial

An expression consisting of at least two terms.

Prime Factor

A factor of an expression that is prime. That is, it has no factors other than itself and 1.

Quadrants

The four regions of the Cartesian plane, which are separated from each other by the coordinate axes. In the first quadrant, $x > 0$ and $y > 0$. In the second quadrant, $x < 0$ and $y > 0$. In the third quadrant, $x < 0$ and $y < 0$. In the fourth quadrant, $x > 0$ and $y < 0$.

Quadratic Equation

An equation of the form $ax^2 + bx + c = 0$ (the highest power is 2).

Ratio

A comparison of two numbers expressed by dividing one by the other.

Reciprocal

The number 1 divided by a given number.

Sense

The direction of an inequality. The sense is preserved if the inequality is multiplied by a positive real, but it is reversed if the inequality is multiplied by a negative real.

Solution

A set of values for the variables in an equation or inequality (one value per variable) that makes the equation or inequality true.

Solution Set

The totality of solutions to a given equation or inequality.

Term

An expression involving variables, exponents, and coefficients.

Trinomial

An expression consisting of three terms.

Universal Set

The set containing all elements under consideration.

Variable

A letter used to represent an arbitrary number.